A Refutation of Popular Diagonalization Applications

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June 24, 2024

Abstract

This paper analyzes the use of diagonalization as applied by Georg Cantor and Alan Turing and shows how their claims do not logically follow from their respective proofs. In Cantor's work, diagonalization and contradiction are used to prove the set of infinite binary sequences does "not have the power of the number-sequence 1, 2, 3, ..., v, ..." *i.e.*, the set is not enumerable. In Turing's work, the diagonal process and contradiction are used to prove the computable sequences are not enumerable. Both proofs incorrectly use contradiction, due to the fact that they assume the truth of two statements and recognize only one. Namely, they assume (1) the set in question is enumerable and (2) the proposed sequence is possible. It is seen that neither proof can concretely and definitively disprove the first assumption as they heavily rely on the second assumption.

Introduction

There are long accepted applications of diagonalization and contradiction that are used to prove some sets are not enumerable. In [2], Cantor uses diagonalization and contradiction to prove the claim that the set of infinite binary sequences does "not have the power of the number-sequence 1, 2, 3, ..., v, ..." *i.e.*, the set is not enumerable. Similarly, in [1], Turing uses the diagonal process (his terminology for diagonalization) and contradiction to prove the claim that the computable sequences are not enumerable.

This paper analyzes both applications and explains how each claim does not logically follow from its proof. I would like to make it clear that this paper does not attempt to prove the opposite of their claims, rather it only attempts to show these claims cannot be definitively concluded based solely on these proofs.

The first two sections analyze Cantor's and Turing's applications of diagonalization and proof by contradiction and show how they incorrectly apply contradiction. The final section has two parts. First, it provides a series of example "proofs" which are similar in nature and incorrect for the same reasons as the referenced applications. Next, it provides additional comments on these applications to aid in gaining an understanding.

Cantor's Claim

This section analyzes Georg Cantor's application of diagonalization and contradiction used to prove the claim that the set of infinite binary sequences is not enumerable as seen in [2]. His proof follows:

> [If] m and w are any two mutually exclusive characters... we consider a set... M of elements
>
> $E = (x_1, x_2, ..., x_v, ...)$
>
> which depend on infinitely many coordinates $x_1, x_2, ..., x_v, ...$ where each of these coordinates is either m or w. Let M be the totality of all elements E.
>
> I now maintain that such a manifold M does not have the power of the sequence $1, 2, ..., v, ...$ [M is not enumerable].
>
> This follows from the following proposition:
>
> 'If $E_1, E_2, ..., E_v, ...$ is any simply infinite... sequence of elements of the manifold M, then there is always an element E_0 of M which corresponds to no E_v.'
>
> For proof, let
>
> $E_1 = (a_{1,1}, a_{1,2}, a_{1,v}, ...)$,
>
> $E_2 = (a_{2,1}, a_{2,2}, a_{1,v}, ...)$,
>
> ....
>
> $E_\mu = (a_{\mu,1}, a_{\mu,2}, a_{\mu,v}, ...)$,
>
> ....

> Here the $a_{\mu,v}$ are determinately m or w. We now define a sequence $b_1, b_2, b_v, ...$, such that b_v is equal to m or w but is different from $a_{v,v}$... If we then consider the element
>
> $E_0 = (b_1, b_2, b_3, ...)$
>
> of M one sees at once the equation
>
> $E_0 = E_\mu,$
>
> can be fulfilled by no integral value of μ, since otherwise for the μ in question and for all integral values of v,
>
> $b_v = a_{\mu,v},$
>
> and so in particular we would have $b_\mu = a_{\mu,\mu}$, which is excluded by the definition of b_v. From this proposition it follows immediately that the totality of elements of M cannot be brought into the sequential form:
>
> $E_1, E_2, ..., E_v, ...;$
>
> otherwise, we would have the contradiction that... E_0 would be an element of M as well as not an element of M.

This claim does not logically follow from the proof because Cantor makes two assumptions to derive the contradiction: (1) the elements of M are enumerable and (2) the description of E_0 produces a valid sequence. Since Cantor makes two assumptions, it cannot be concluded that the enumeration of M allows for the contradiction.

The described sequence, E_0, will never be possible; not necessarily because M is not enumerable, but because E_0 is simply described an impossible sequence. Recall the v-th figure of E_0 is described as the opposite of the v-th figure of the v-th element in the enumeration of M. Since E_0 is described as an element of M, it exists at some position, say μ. By the definition of E_0, the μ-th figure of E_0 is the opposite of the μ-th figure of E_0. This is similar to the following example: Let A be an infinite binary sequence whose n-th figure is the opposite of the n-th figure of A. Both A and E_0 are simply impossible sequences.

Turing's Claim

This section analyzes Alan Turing's application of the diagonal process and contradiction used to prove the claim that the computable sequences are not enumerable as seen in [1]. His proof follows:

> Or we might apply the diagonal process. "If the computable sequences are enumerable, let α_n be the n-th computable sequence, and let $\phi_n(m)$ be the m-th figure in α_n. Let β be the sequence with $1-\phi_n(n)$ as its n-th figure. Since β is computable, there exists a number K such that $1-\phi_n(n) = \phi_K(n)$ [for] all n. Putting $n = K$, we have $1 = 2\phi_K(K)$, *i.e.* 1 is even. This is impossible. The computable sequences are therefore not enumerable".

This claim does not logically follow from the proof because Turing makes two assumptions to derive the contradiction: (1) the computable sequences are enumerable and (2) β is computable. Since Turing makes two assumptions, it cannot be concluded that the enumeration of the computable sequences allows for the contradiction. Turing recognizes the second assumption as he writes:

> The fallacy in this argument lies in the assumption β is computable. It would be true if we could enumerate the computable sequences by finite means... The simplest and most direct proof of this is by showing that, if this general process [equivalent to enumerating the computable sequences] exists, then there is a machine which computes β.

I would like to point out that the description of β will *never* produce a valid sequence. Not necessarily because the computable sequences are not enumerable, but because β is simply described as an impossible sequence. Recall we assume β is computable and the computable sequences are enumerable. Therefore β exists in the enumeration, say at position K. The K-th figure of β is therefore $1-$ the K-th figure of β. This is similar to the following example: Let A be an infinite binary sequence whose n-th figure is the opposite of the n-th figure of A. Both A and β are simply impossible sequences.

General Scenario

This section presents a general scenario with a description of an impossible sequence and proof by contradiction similar to the referenced applications.

Let p be a statement whose truth is in question. Let q be a description of an object (a set, sequence, etc. or anything that can derive a contradiction through self reference) as follows: "if p is true, construct an impossible object, otherwise if p is false, construct a possible object." We then apply a proof by contradiction: Suppose p is true. Then by using q, we can construct an impossible object and derive a contradiction. Therefore p must be false.

It is seen here that the truth of p does not matter and the proof depends nearly entirely on the description of q. This is similar to the referenced applications in that all the proofs take the following steps: (1) assume the truth of the statement in question, (2) propose an impossible sequence based on the statement in question (and assume this sequence is possible), (3) derive a contradiction, and (4) conclude the first statement is false.

This feels like an attractive proof by contradiction because assuming the truth of the statement in question allows for a contradiction while assuming the falsehood of the statement in question does not. However, this is only the case because the second assumption (that the sequence is possible) does not exist without assuming the truth of the statement in question (due to the fact that the sequence cannot even be defined without assuming the truth of statement in question); therefore, any problems in either assumption will be alleviated upon assuming the falsehood of the statement in question.

The key to understanding this is recognizing that these proofs make the additional assumption that the proposed sequence is possible. Recognizing this, it is seen that this second assumption can allow for the contradiction and the truth of first assumption cannot be held entirely responsible for the contradiction. Since the first assumption cannot be held entirely responsible for the contradiction, the claims in the referenced applications do not logically follow from their respective proofs.

Examples and Additional Comments

This section provides examples of incorrectly applying proof by contradiction as well as additional comments to bridge the gap between the general scenario and the referenced applications. To repeat the problem at hand, the following examples work similarly to the referenced applications in that they (1) assume the truth of the statement in question, (2) propose an impossible sequence based on the statement in question (and assume this sequence is possible), (3) derive a contradiction, and (4) conclude the first statement is false.

Example 1

Suppose there exists a positive integer, K, which is fixed, yet arbitrarily large. Construct an infinite binary sequence, B, such that the n-th figure of B is 1 except for the K-th figure of B, which is the opposite of the K-th figure of B. The K-th figure of B cannot be 1, because then the K-th figure of B must be 0. Similarly, the K-th figure of B cannot be 0 because then the K-th figure of B must be 1. This is a contradiction. Therefore a fixed, yet arbitrarily large number cannot exist.

Example 2

Let M be the set of infinite binary sequences. Assume there exists a set, P, of all possible subsets of M where each set in P contains a single element of M. Let P_C be a set in P containing the infinite binary sequence C, as yet to be defined. Since there is only one element in P_C, enumerate the set with $i = 1$. Let C be an infinite binary sequence of 1's except the i-th figure in C is the opposite of the i-th figure of the i-th element in P_C. Since C is an infinite binary sequence, it is in M. Since C is in M, it exists in one of the sets in P. Therefore we can refer to the set containing C as P_C. The i-th element of C will always be the opposite of the i-th element of C. This is impossible. Therefore there cannot exist a set of all possible subsets of M where each subset contains a single element of M.

Additional Comments

This subsection contains additional comments regarding the proofs seen thus far in order to aid in gaining an understanding.

Given the current assumptions of the laws of the universe, we can imagine the description of a sequence, A, where the n-th figure of A is the opposite of the n-th figure of A. Naturally, we do not conclude we have assumed the truth of a false statement: we can clearly see the description of A is problematic and produces an impossible sequence.

Consider one of the referenced applications, say the set of infinite binary sequences is enumerable. In an environment where this statement is assumed to be a true fact of the universe, it would be natural to see the proposed description does not produce a possible sequence. Just as above we conclude the description of A is invalid rather than concluding we have assumed the truth of a false statement, here we naturally conclude the proposed description is invalid rather than concluding the first assumption (the set of binary sequences is enumerable) is false.

Conclusion

It is seen the proposed applications of diagonalization and proof by contradiction improperly use contradiction due to the fact that they assume the truth of two statements and recognize only one. It seems it is especially easy to fall into this trap when applying diagonalization since both of the referenced applications use diagonalization to construct an object and unknowingly make the assumption this object is valid. The definition of this object is dependent on the statement in question so problems in either assumption are alleviated when the original statement is assumed to be false (here the second assumption cannot even be defined without the first assumption). Since two assumptions are made, it cannot be said that the first assumption (that the set in question is enumerable) alone allows for the derivation of the contradiction. Applications of diagonalization and proof by contradiction done in the same way as the referenced applications are incorrect and do not allow for conclusions similar to those claimed.

References

[1] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 2(42):230–265, 1936.

[2] Ewald William Bragg. *From Kant to Hilbert Volume 2.* Number Volume II in From Kant to Hilbert. OUP Oxford, 2005.